NUMERICAL-ANALYTICAL METHOD OF CALCULATING THE JET FLOW OF A CURRENT OF HEAVY FLUID AROUND CURVED OBSTACLES

L. G. Guzevskii

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An approximative method is proposed for solving problems for plane stationary currents of an ideal incompressible fluid with one free boundary. The problem of the flow of a heavy fluid out from under a curved panel and symmetrical problems on the cavitation flow around curved arcs according to different classical systems [1] in a longitudinal gravitational field belong to this class, for example. The method is illustrated on the example of the solution of the problem of cavitation flow around a fixed curved arc mounted on a horizontal base by Ryabushinskii's system in a gravitational field perpendicular to the direction of the inflowing current.

The fixed curved boundary (the arc P) is given by the equation $\Psi = \Psi(l)$, where Ψ is the angle of inclination of the tangent to the given arc to a horizontal line and l is the abscissa of the arc. The arc P is approximated by some curve $\Psi_* = \Psi_*(l)$ having a continuous derivative in such a way that $\Psi_* = \Psi$ for a finite number of values of the arc abscissa l. The linearization of the boundary conditions for the free surface, in which the condition of constant pressure in the cavity is satisfied exactly at a finite number of points of this boundary, is conducted as in [2, 3]. With such a finite approximation of the boundary condition the solution of the problem (in the plane of the auxiliary parametric variable) is obtained in explicit form. The principal difficulty consists in the determination of numerical values for the parameters entering into the solution of the problem from a system of nonlinear transcendental equations. The method of solving this system using an electronic computer is based on an algorithm presented in [3] for systems of equations of this type.

A study of questions of the existence and uniqueness of the solution for a broad class of problems of the hydrodynamics of an ideal fluid with free boundaries based on the method of finite approximation is conducted in the works of V. N. Monakhov [2]. In this case the curved boundaries are approximated by polygons.

Previously the approximative analytical solutions of problems on the jet flow around fixed curved obstacles were obtained only in the case of a weightless fluid [4].

Let us examine the plane established flow of an ideal heavy fluid running up against a curved arc P which is located on a horizontal straight base. As the diagram of the cavitation flow we adopt Ryabushinskii's diagram with a "mirror" (Fig. 1). Because of the assumed symmetry of the flow it is sufficient to limit the examination to the current in the region D_z bounded by the flow line CDAB and the equipotential line BC. We set the origin of the Cartesian coordinate system at the point A of breaking away of the flow; the abscissa is directed horizontally and the ordinate perpendicular to the velocity vector of the inflowing current.

The curved arc P is given by the equation

$$\Psi = \Psi(l_*) \qquad (l_* = l/L, \quad 0 \leqslant l_* \leqslant 1) \tag{1}$$

where Ψ is the angle between the tangent to the arc P and the x axis; l is the length of the arc measured along P; L is the total length of the arc P. Henceforth we will assume that the function $\Psi(l_*)$ satisfies the Hölder condition.

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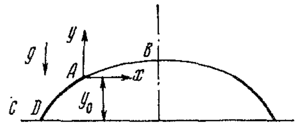

Fig. 1

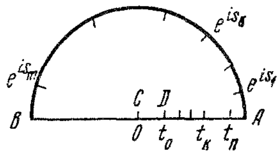

Fig. 2

The unknown (free) boundary AB is sought from the condition of constant pressure in the chamber, which on the basis of Bernoulli's equation has the form

$$\frac{q^2}{q_0^2} = 1 - \frac{2}{p(1+\sigma)\,\mathrm{Fr}^2}\frac{y}{L} \qquad \left(\sigma = \frac{q_0^2}{q_\infty^2} - 1,\ \mathrm{Fr} = \frac{q_\infty}{\sqrt{gy_0}}\right) \tag{2}$$

Here $q = q(y)$ is the velocity at the point of the free boundary with the ordinate y, $q_0 = q(0)$, and q_∞ is the velocity of the undisturbed flow; $p = y_0/L$ is fixed parameter characterizing the geometry of the obstacle; y_0 is the vertical dimension of the obstacle; σ is the cavitation number, Fr is the Froude number, and g is the acceleration of gravity.

The region D_z is transformed conformally onto the inner part of a unit semicircle Γ_ζ

$$|\zeta| < 1, \qquad \mathrm{Im}\,\zeta > 0$$

in accordance with the points shown in Fig. 2 in which the free boundary converts into the arc of the circle $\zeta = e^{is}$ $(0 \le s \le \pi)$, while the remaining part of the boundary converts into the real diameter $\zeta = t$ $(-1 \le t \le 1)$. Then the derivative of the function which performs the transformation to Γ_ζ on the region in the plane of the complex potential $w = \varphi + i\psi$ takes the form

$$\frac{dw}{d\zeta} = Kq_0\frac{1-\zeta}{\zeta^{3/2}} \tag{3}$$

where K is some real constant.

Let us introduce Zhukovskii's function

$$\Omega(\zeta) = i\ln\frac{1}{q_0}\frac{dw}{dz} \equiv \vartheta(\zeta) + i\tau(\zeta) \tag{4}$$

where ϑ is the angle between the velocity vector and the x axis and $dw/dz = qe^{-i\vartheta}$ is the complex flow velocity.

It is easy to obtain a representation for the function $\Omega(\zeta)$, analytical in Γ_ζ, through the functions $\vartheta(t)$ and $\tau(s) = \ln[q(s)/q_0]$, where $\vartheta(t)$ is the limiting value of the real part of Ω for $\zeta \to t \in [t_0, 1]$, while $\tau(s)$ is the limiting value of the imaginary part of Ω for $\zeta \to e^{is}$ $(0 \le s \le \pi)$. In fact, at the boundary of the semicircle Γ_ζ the function $\Omega(\zeta)$ satisfies the following boundary conditions:

$$\mathrm{Im}\,\Omega(e^{is}) = \tau(s) = \ln(q(s)/q_0), \qquad s \in [0, \pi]$$

$$\mathrm{Re}\,\Omega(t) = \begin{cases} \vartheta(t), & t \in [t_0, 1] \\ 0, & t \in [-1, t_0] \end{cases}$$

Hence, as a result of solving the composite boundary problem we obtain a representation for the function

$$\Omega(\zeta) = \frac{1-\zeta^2}{\pi i}\left[\int_{t_0}^{1}\frac{\vartheta(t)\,dt}{(t-\zeta)(1-\zeta t)} - \int_0^{\pi}\frac{\tau(s)\,ds}{1-2\zeta\cos s+\zeta^2}\right] \tag{5}$$

According to Eqs. (3) and (4) the relation between the region of flow and the parametric plane is given by the function

$$z(\zeta) = K\int_1^{\zeta}\frac{1-\zeta}{\zeta^{3/2}}e^{i\Omega(\zeta)}\,d\zeta \tag{6}$$

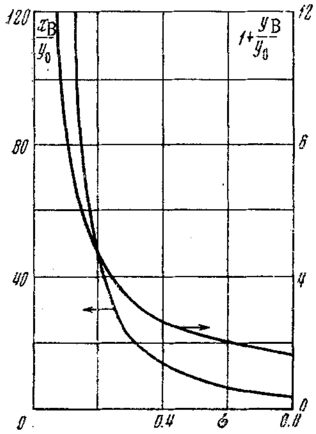

Fig. 3

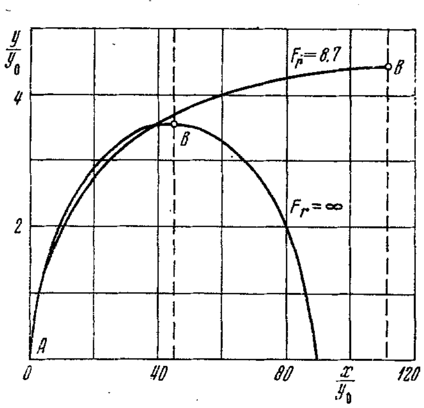

Fig. 4

with $\Omega(\zeta)$ determined by Eq. (5). The dependence $l = l(t)$ is found from the equation

$$l(t) = K\int_{t_0}^{t} \frac{1-t}{t^{3/2}} e^{-\operatorname{Im}\Omega(t)}\, dt \tag{7}$$

where $\operatorname{Im}\Omega(t)$ is the limiting value of the imaginary part of $\Omega(\zeta)$ for $\zeta \to t \in [-1,1]$, where according to Eq. (5)

$$\operatorname{Im}\Omega(t) = -\frac{1-t^2}{\pi}\left[\int_{t_0}^{1} \frac{\vartheta(\xi)\, d\xi}{(\xi - t)(1 - t\xi)} - \int_{0}^{\pi} \frac{\tau(s)\, ds}{1 - 2t\cos s + t^2}\right] \tag{8}$$

For arbitrarily given $\vartheta(t)$, $t_0 \le t \le 1$ and $\tau(s)$, $0 \le s \le \pi$ Eqs. (4)-(6) give a general solution of some inverse problem on the flow around an arc whose form is not known beforehand, while at the boundary AB the pressure, generally speaking, differs from a constant pressure. To solve the direct problem on constructing the cavitation flow around a given obstacle P it is necessary to satisfy the boundary condition (2) at the free boundary and the relation $\vartheta(t) = \Psi[l(t)/l(1)]$ resulting from the specification of the given arc P through Eq. (1)

$$\tau(s) = \frac{1}{2}\ln\left[1 - \frac{2\lambda}{J}\int_{0}^{s} \sin\frac{s}{2}\, e^{-\tau(s)} \sin\{\operatorname{Re}\Omega(e^{is})\}\, ds\right], \quad s \in [0, \pi] \tag{9}$$

$$\vartheta(t) = \Psi\left[\frac{1}{J}\int_{t_0}^{t} \frac{1-t}{t^{3/2}} e^{-\operatorname{Im}\Omega(t)}\, dt\right], \quad t \in [t_0, 1]$$

In these equations

$$\lambda = \frac{2}{p(1+\sigma)\,\mathrm{Fr}^2}, \quad J = \frac{l(1)}{K} = \int_{t_0}^{t} \frac{1-t}{t^{3/2}} e^{-\operatorname{Im}\Omega(t)}\, dt$$

$$\operatorname{Re}\Omega(e^{is}) = \frac{\sin s}{\pi}\left[2\int_{t_0}^{1} \frac{\vartheta(t)\, dt}{1 - 2t\cos s + t^2} - \int_{0}^{\pi} \frac{\tau(\gamma)\, d\gamma}{\cos\gamma - \cos s}\right] \tag{10}$$

The equation for the cavitation number follows from Eqs. (4) and (5) at $\zeta = 0$:

$$\sigma = \frac{q_0^2}{q_\infty^2} - 1 = \exp\left\{\frac{2}{\pi}\left[\int_{t_0}^{1} \frac{\vartheta(t)}{t}\, dt - \int_{0}^{\pi} \tau(s)\, ds\right]\right\} - 1$$

which can serve as an additional condition for determining the parameter t_0. However, we will take t_0 as fixed from now on. Consequently, the cavitation number is determined after solving the problem.

With the goal of an approximate determination of the functions $\vartheta(t)$ and $\tau(s)$ from the system (9) let us begin as follows. We introduce into the examination a number of fixed points distributed on the boundary of the semicircle Γ_ζ:

$$\zeta_k = e^{is_k} \quad (0 = s_0 < s_1 < \ldots < s_m < s_{m+1} = \pi), \quad t_0 < t_1 < \ldots < t_n < t_{n+1} = 1$$

Then we set

$$\tau(s) = \sum_{k=0}^{m} \tau^k(s), \qquad \vartheta(t) = \sum_{k=0}^{n} \vartheta^k(t) \tag{11}$$

where

$$\tau^k(s) = \begin{cases} 0, \ s \text{ outside} [s_k, s_{k+1}] \\ \tau(s_k) + \dfrac{\cos s - \cos s_k}{\cos s_{k+1} - \cos s_k} [\tau(s_{k+1}) - \tau(s_k)] \equiv a_k + b_k \cos s, \quad s \in [s_k, s_{k+1}] \end{cases}$$

$$\vartheta^k(t) = \begin{cases} 0, \ t \text{ outside} [t_k, t_{k+1}] \\ \vartheta(t_k) + \dfrac{t - t_k}{t_{k+1} - t_k} [\vartheta(t_{k+1}) - \vartheta(t_k)] \equiv c_k + d_k t, \quad t \in [t_k, t_{k+1}] \end{cases}$$

From the construction

$$\tau^k(s_k) = \tau(s_k) \equiv \tau_k \quad (k = 0, \ldots, m+1), \quad \vartheta^k(t_k) = \vartheta_k \quad (k = 0, \ldots, n+1).$$

In such an interpolation of the functions $\vartheta(t)$ and $\tau(s)$ sought, the integrals entering into the representation (5) of the function $\Omega(\zeta)$ are calculated in explicit form:

$$\Omega(\zeta) = \frac{1}{\pi i}\left\{\sum_{k=0}^{m} b_k (s_{k+1} - s_k) \frac{1-\zeta^2}{2\zeta} - i\left(a_k + b_k \frac{1+\zeta^2}{2\zeta}\right)[F_k(\zeta) - F_{k+1}(\zeta)] + R(\zeta) - R\left(\frac{1}{\zeta}\right)\right\} \tag{12}$$

where

$$F_k(\zeta) = \ln \frac{\zeta - e^{is_k}}{1 - \zeta e^{is_k}}, \quad R(\zeta) = \sum_{k=0}^{n} (c_k + d_k \zeta) \ln \frac{t_{k+1} - \zeta}{t_k - \zeta}$$

Satisfying Eq. (9) at the points $\zeta_k = e^{is_k}$ $(k = 1, \ldots, m+1)$ and t_k $(k = 1, \ldots, n)$ we arrive at a system of $m+n+1$ equations relative to the parameters $\tau_1, \ldots, \tau_{m+1}, \vartheta_1, \ldots, \vartheta_n$

$$\begin{gathered} \tau_k = \frac{1}{2} \ln\left[1 - \frac{2\lambda}{J} \int_0^{s_k} \sin\frac{s}{2} e^{-\tau(s)} \sin \vartheta(e^{is}) \, ds\right] \quad (k = 1, \ldots, m+1) \\ \vartheta_k = \Psi\left[\frac{1}{J} \int_{t_0}^{t_k} \frac{1-t}{t^{3/2}} e^{-\tau(t)} \, dt\right] \quad (k = 1, \ldots, n) \\ (\tau(t) = \operatorname{Im} \Omega(t), \quad -1 \leqslant t \leqslant 1, \quad \vartheta(e^{is}) = \operatorname{Re} \Omega(e^{is}), \quad 0 \leqslant s \leqslant \pi) \end{gathered} \tag{13}$$

After finding the solution of the given system of equations the flow is determined by Eqs. (3), (6), and (12). In this case (from the construction) Eq. (1) of the arc P will be satisfied at the points $z(t_k)$, $k = 0, \ldots, n+1$, while the condition of constant pressure (2) will be satisfied at the boundary AB: at the points $z(e^{is_k})$, $k = 0, \ldots, m+1$.

In the first approximation the problem is solved for $m = n = 0$. The unknown parameter $\tau_1 = \ln(q_B)/q_0)$ is determined from the single equation which remains from the system of equations (13) in this case.

Let us now assume that the problem is solved in the i-th approximation. Then for some fixed parameters s_k^i $(k = 1, \ldots, m_i)$ and t_k^i $(k = 1, \ldots, n_i)$ the values of ϑ_k^i $(k = 1, \ldots, n_i)$ and τ_k^i $(k = 1, \ldots, m_i + 1)$ corresponding to them are determined from system (13). Taking into account the form of the functions $\vartheta(t)$ and $\tau(s)$ given by Eqs. (11) it is easy to construct the inverse functions $t = T(\vartheta)$ and $s = S(\tau)$.

The solution of the problem in the $(i+1)$-th approximation with $m = m_{i+1} > m_i$ and $n = n_{i+1} > n_i$ is conducted in the following sequence. First, using the functions $t = T(\vartheta)$ and $s = S(\tau)$ constructed earlier, the parameters $t_k = t_k^{i+1}$ and $s_k = s_k^{i+1}$ are chosen:

$$t_k = T(\Psi_k), \quad \Psi_k = \Psi\left(\frac{k}{n_{i+1}+1}\right) \quad (k = 1, \ldots, n_{i+1})$$
$$s_k = S(f_k), \quad f_k = \frac{k}{m_{i+1}+1}\tau^{i}_{m_{i+1}} \quad (k = 1, \ldots, m_{i+1}+1) \tag{14}$$

The system of equations (13) is solved by the method of iterations. For this the initial approximation $\vartheta_k^{(0)}$, $\tau_k^{(0)}$ is given in the following way:

$$\vartheta_k^{(0)} = \Psi_k \quad (k = 1, \ldots, n_{i+1}), \quad \tau_k^{(0)} = f_k \quad (k = 1, \ldots, m_{i+1}+1)$$

In choosing the nodes of interpolation of the functions $\vartheta(t)$ and $\tau(s)$ from the function (14) it can be expected that the images of the points t_k and e^{is_k} in the transformation (6) will be distributed rather evenly along the boundary DAB. After finding the solution of the system a test is made of the accuracy of the approximative solution obtained for the hydraulic problem:

1) the resulting profile DA is compared with the given curved arc P;

2) the shapes of the cavity in the i-th and (i+1)-th approximations are compared and the condition of constant pressure is checked over the entire free boundary.

The process described for constructing an approximative solution of the original problem is repeated until Eqs. (1) and (2) are satisfied with the required precision.

Calculations conducted for a number of curved profiles have shown the high effectiveness of the proposed method.

The results of a calculation of the problem of cavitation flow around an obstacle in the shape of an arc of a circle with a tear-away angle of $\Psi(1) = 35°$ are presented below as an example. Curves of the dependence of the cavity dimensions on the cavitation number in the case of a weightless fluid are shown in Fig. 3. For n = 15 the maximum deviation from the arc of a circle in the shape of the profile obtained is not more than 0.2% (comparison by radius in polar coordinates) and the computation time for one variant of the problem on a BÉSM-6 machine does not exceed one minute. For n = 31 the error becomes less than 0.05%.

A comparison of the shape of the free boundary for flows of a weightless and a heavy fluid with $\sigma = 0.2$ is presented in Fig. 4. In this case the calculations were conducted with m = n = 15 and the error does not exceed 0.2%.

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